

B -submodules of $\mathfrak{g}/\mathfrak{b}$ and Smooth Schubert Varieties in G/B

JAMES B. CARRELL

Abstract Let G be a semisimple linear algebraic group over \mathbb{C} without G_2 factors, B a Borel subgroup of G and $T \subset B$ a maximal torus. The flag variety G/B is a projective G -homogeneous variety whose tangent space at the identity coset is isomorphic, as a B -module, to $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{b} = \text{Lie}(B)$. Recall that if w is an element of the Weyl group W of the pair (G, T) , the Schubert variety $X(w)$ in G/B is by definition the closure of the Bruhat cell BwB . In this note we prove that $X(w)$ is nonsingular if and only if the following two conditions hold: 1) its Poincaré polynomial is palindromic and 2) the tangent space $TE(X(w))$ to the set T -stable curves in $X(w)$ through the identity is a B -submodule of $\mathfrak{g}/\mathfrak{b}$. This gives two criteria in terms of the combinatorics of W which are necessary and sufficient for $X(w)$ to be smooth: $\sum_{x \leq w} t^{\ell(x)}$ is palindromic, and every root of (G, T) in the convex hull of the set of negative roots whose reflection is less than w (in the Bruhat order on W) has the property that its T -weight space (in $\mathfrak{g}/\mathfrak{b}$) is contained in $TE(X(w))$. However, these conditions don't characterize the smooth Schubert varieties when G has type G_2 .

1. INTRODUCTION

Schubert varieties in the flag variety of a linear algebraic group G were originally defined by Chevalley in a famous unpublished paper [9], where it was remarked offhandedly that all Schubert varieties were probably smooth. This was an oversight, of course, because it was already known that there exist singular Schubert varieties in Grassmannians. The question of actually determining the smooth Schubert varieties in an arbitrary flag variety has subsequently been treated in many places. For example, when G is of type A , Lakshmibai and Seshadri [14] computed the tangent spaces of all the Schubert varieties, and subsequently Deodhar [10] used this to show that in type A , the rationally smooth Schubert varieties are all smooth. Dale Peterson showed this result holds whenever G is simply laced (see [8] for a proof). In [13], Lakshmibai and Sandhya showed that, in type A , the smooth Schubert varieties are exactly the ones whose defining permutation avoids a certain pattern, and, in [5], Billey and Postnikov extended pattern avoidance to all classical G . Other results concerning globally smooth Schubert varieties in G/B in various settings include [1], [4], [8], [12], [15], [16]. The main result of this note extends Peterson's criterion for smoothness to the non-simply laced setting by adding a condition which is vacuously satisfied in the simply laced setting.

2. PRELIMINARY REMARKS

Before stating our result, we will review some elementary definitions and well known facts about Schubert varieties. Let G denote a semisimple linear algebraic group over \mathbb{C} with a fixed Borel subgroup B and maximal

torus $T \subset B$, and let $\mathfrak{g}, \mathfrak{b}$ and \mathfrak{t} denote their respective Lie algebras. Recall the Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

of \mathfrak{g} into T -weight spaces, where Φ is the root system of the pair (G, T) . The set of positive roots consists of the roots corresponding to the T -module \mathfrak{b} . This set is denoted by Φ^+ . One has $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^- = -\Phi^+$. Thus, the set of T -weights on $\mathfrak{g}/\mathfrak{b}$ is Φ^- . The flag variety G/B of G is a G -homogeneous projective variety, and it is well known its tangent space $T_e(G/B)$ at the identity coset $e = B/B$ is isomorphic with $\mathfrak{g}/\mathfrak{b}$. Note, that the identity coset in G/B is fixed by B , so $T_e(G/B)$ is in fact a B -module, and, by considering the B -equivariant projection $\pi : G \rightarrow G/B$, one obtains that $T_e(G/B)$ and $\mathfrak{g}/\mathfrak{b}$ are isomorphic as B -modules. We will henceforth make the identification

$$T_e(G/B) = \mathfrak{g}/\mathfrak{b} = \sum_{\alpha < 0} \mathfrak{g}_{\alpha}.$$

Let $W = N_G(T)/T$ be the Weyl group of (G, T) , and recall that W is a finite reflection group generated by the reflections r_{α} (of \mathfrak{h}) through roots α . By the Bruhat decomposition $G = BWB$, G/B is the union of the B -orbits of the cosets $wB = n_w B$ as w ranges over W , where $n_w \in N_G(T)$ is a representative of w . One of the basic results of [9] says that the B -orbit $Bw \subset G/B$ is isomorphic to affine space $\mathbb{C}^{\ell(w)}$, where ℓ is the length function on W . The Zariski closure $X(w)$ of the B -orbit Bw is called the Schubert variety associated to w . Thus $\dim X(w) = \ell(w)$. Another basic result of [9] is that the Bruhat order \leq on W , defined combinatorially in terms of reflections, is compatible with the natural geometric order on the B -orbits: $x \leq w$ if and only if $X(x) \subseteq X(w)$. Moreover, the T -fixed point set $X(w)^T$ is precisely the set $\{x \in W \mid x \leq w\}$. It follows from these remarks that the Poincaré polynomial of $X(w)$, with respect to ordinary rational homology, has the well known expression

$$(1) \quad P(X(w), t) = \sum_{x \leq w} t^{2\ell(x)}.$$

3. STATEMENTS OF RESULTS

The purpose of this paper is to give a simple constructive criterion which describes which Schubert varieties with palindromic Poincaré polynomial in G/B are smooth that holds unless G has a G_2 factor and, in fact, is false in G_2/B . The condition that the Poincaré polynomial of a Schubert variety $X(w)$ is palindromic is equivalent to the more difficult to formulate condition that $X(w)$ is rationally smooth (cf. [6]). A variety is said to be rationally smooth at a point if it satisfies local Poincaré duality at the point and globally rationally smooth if it is rationally smooth at every point [11]. We will state the smoothness criterion in three successive ways. The first

involves only the linear span $\Theta(w) \subset T_e(X(w))$ of the reduced tangent cone to $X(w)$ at the identity element.

Theorem 1. *Suppose G has no G_2 factors and $X(w)$ is a Schubert variety in G/B whose Poincaré polynomial is palindromic (i.e. $X(w)$ is rationally smooth). Then $X(w)$ is smooth if and only if $\dim \Theta(w) = \ell(w)$.*

In type A , it follows readily from a result of Lakshmibai and Seshadri [14] that $\Theta(w) = T_e(X(w))$. By combining a result of the author [7] and Polo [15], this equality also holds in type C . Therefore, Theorem 1 follows easily in types A and C from the Borel Fixed Point Theorem. Indeed, since $X(w)$ is B -stable and the identity coset e is fixed by B , $X(w)$ is smooth if and only if it is smooth at e .

The second formulation of the smoothness criterion requires that we bring in the tangent space $TE(X(w))$ to the set $E(X(w))$ of T -curves to $X(w)$ at e and discuss its relationship with $\Theta(w)$. Unless otherwise stated, proofs of the assertions here are in [6]. Let

$$TE(X(w)) = \sum_{C \in E(X(w))} T_e(C).$$

Each T -curve in G/B is smooth, and hence if $C \in E(X(w))$, $T_e(C) = \mathfrak{g}_\alpha$ for some $\alpha < 0$. One also has that $\dim TE(X(w)) = |E(X(w))|$.

Lemma 1. *Let $\Phi(w)$ denote the set of $\alpha < 0$ such that T has weight α on $T_e(C)$ for some $C \in E(X(w))$. Then*

$$\Phi(w) = \{\alpha < 0 \mid r_\alpha \leq w\}.$$

Thus

$$(2) \quad TE(X(w)) = \sum_{\alpha \in \Phi(w)} \mathfrak{g}_\alpha \subseteq \Theta(w).$$

By Deodhar's inequality, $|E(X(w))| \geq \ell(w)$, so $\dim TE(X(w)) \geq \ell(w)$. It follows that $\dim \Theta(w) \geq \ell(w)$ with equality if and only if $TE(X(w)) = \Theta(w)$. Finally, if $X(w)$ is rationally smooth at e , then $\dim TE(X(w)) = \ell(w)$. However, knowing $\dim TE(X(w)) = \ell(w)$ does not guarantee that $X(w)$ is rationally smooth at e .

Note that if G is simply laced, then Theorem 1 is vacuously true since a result of Dale Peterson's says that every rationally smooth Schubert variety in G/B is smooth (see [8] for a discussion and proof). The assumption that $\dim \Theta(w) = \ell(w)$ is automatically guaranteed when $X(w)$ is rationally smooth, since $TE(X(w)) = \Theta(w)$ for all w in the simply laced case. In other words, Theorem 1 gives the additional condition under which Peterson's result holds for the non G_2 setting.

The second formulation of Theorem 1 doesn't involve $\Theta(w)$. Since $X(w)$ is B -stable and the identity coset e is fixed by B , $T_e(X(w))$ and $\Theta(w)$ are B -stable submodules of $\mathfrak{g}/\mathfrak{b}$. On the other hand, $TE(X(w))$ in general isn't B -stable.

Theorem 2. *Suppose G has no factors of type G_2 , and assume the Poincaré polynomial of the Schubert variety $X(w)$ in G/B is palindromic, i.e. $X(w)$ is rationally smooth. Then $X(w)$ is smooth if and only if $TE(X(w))$ is a B -submodule of $\mathfrak{g}/\mathfrak{b}$.*

The proof of this version uses the result that $\Theta(w)$ is the B -module span of $TE(X(w))$ (see [7, Theorem X]). Hence if $X(w)$ is rationally smooth at e and $TE(X(w))$ is a B -submodule of $\mathfrak{g}/\mathfrak{b}$, then clearly $\dim \Theta(w) = \ell(w)$.

The third formulation uses a description of the weights occurring in $\Theta(w)$. This description involves the following convexity condition. Note that here, the root system Φ is assumed to lie in the dual space \mathfrak{t}^* .

Theorem 3. *Let $w \in W$ be arbitrary, and suppose $\mathcal{H}(w)$ is the convex hull of $\Phi(w)$ in \mathfrak{t}^* over \mathbb{R} . Then*

$$(3) \quad \Theta(w) = \sum_{\alpha \in \mathcal{H}(w) \cap \Phi^-} \mathfrak{g}_\alpha.$$

Thus $TE(X(w)) = \Theta(w)$ if and only if $\Phi(w) = \mathcal{H}(w) \cap \Phi^-$. Moreover, if $TE(X(w))$ is B -stable, then $TE(X(w)) = \Theta(w)$.

Proof. All one needs to do is quote Theorem 3 and Corollary 1 of [7]. \square

Our final formulation of the main result involves only $\Phi(w)$.

Theorem 4. *Suppose G has no factors of type G_2 , and assume the Poincaré polynomial of the Schubert variety $X(w)$ in G/B is palindromic, i.e. $X(w)$ is rationally smooth. Then $X(w)$ is smooth if and only if $\Phi(w) = \mathcal{H}(w) \cap \Phi^-$.*

Let us conclude this introduction with a few remarks about the inclusions

$$TE(X(w)) \subseteq \Theta(w) \subseteq T_e(X(w)).$$

In the ADE setting, $TE(X(w)) = \Theta(w)$. This is due to two facts: first, every T -line in the reduced tangent cone to $X(w)$ at e arises as $T_e(C)$ for a unique $C \in E(X(w))$, and, second, every root is long (hence $\Phi(w) = \mathcal{H}(w) \cap \Phi^-$).

4. PROOFS OF MAIN THEOREMS

It is clear from the above discussion that we only need to prove one version of the main result. Perhaps surprisingly, it turns out that the easiest version to deal with is the B -module version of Theorem 2. We already noted that the main results hold in types $ACDE$, so it suffices to check the B and F_4 cases. We will see below that type B is easy to conclude from a result of Billey. Thus the only sticking point is type F_4 . In order to get around this difficulty, we use the notion of a stellar root system as introduced in [5].

Let us begin by recalling what a stellar root system is. A reduced root system Φ distinct from A_1 and A_2 is called *stellar* if its Dynkin diagram is star shaped. That is, there exists a vertex which is on every edge. Thus the stellar root systems are $B_2, C_2, G_2, A_3, B_3, C_3, D_4$. Let $\mathbb{R}\Phi$ denote the

real subspace of \mathfrak{t}^* generated by Φ . A subroot system of Φ is by definition a subset Ψ of Φ of the form $\Psi = \Phi \cap V$, where V is a subspace of $\mathbb{R}\Phi$. A subroot system is a root system. Given $w \in W$, the *inversion set* of w is the set $I_\Phi(w) = \Phi^+ \cap w(\Phi^-)$. The inversion set $I_\Phi(w)$ uniquely determines w , and for any root subsystem $\Psi = \Phi \cap V$, $I_\Phi(w) \cap V$ is the inversion set of a unique element $v \in W_\Psi$, the Weyl group of Ψ . The *flattening map*

$$fl : W \rightarrow W_\Psi$$

is the assignment $w \rightarrow v$. That is, $fl(w) = v$.

Let Ψ be a subsystem of Φ , and let G_Ψ (resp. B_Ψ) be the subgroup of G generated by T and the root subgroups U_α , where $\alpha \in \Psi$ (resp. the U_α with $\alpha \in \Psi^+$). The following result of Billey and Postnikov classifies the rationally smooth (resp. smooth) Schubert varieties for arbitrary G in terms of stellar subsystems.

Theorem 5. *A Schubert variety $X(w)$ in G/B is rationally smooth (resp. smooth) if and only if for every stellar subsystem Ψ of Φ , the Schubert variety in G_Ψ/B_Ψ corresponding to $fl(w)$ is also rationally smooth (resp. smooth), where (G_Ψ, B_Ψ) is the unique pair of subgroups of (G, B) determined by Ψ .*

Let now prove Theorem 2. Suppose $X(w)$ is a rationally smooth Schubert variety in G/B , and let Ψ be a stellar subsystem of Φ . If $v \in W_\Psi$, let $Y(v)$ denote the corresponding Schubert variety in G_Ψ/B_Ψ . Put $v = fl(w)$. Then $Y(v)$ is rationally smooth. By the discussion in [2, Section 4],

$$Y(v) = X(w) \cap (G_\Psi/B_\Psi).$$

We claim this implies that $TE(Y(v))$ is B_Ψ -stable. To see this, suppose $\alpha \in \Psi^-$ is a weight of $TE(Y(v))$, and suppose that $\beta \in \Psi^+$ is such that $\alpha + \beta$ is also a weight of the tangent space $T_e(Y(v))$. In particular, $\alpha + \beta \in \Psi^-$, so it follows that $\alpha + \beta$ is a weight in $T_e(X(w))$. By assumption, $\alpha + \beta$ is a weight of $TE(X(w))$, since $TE(X(w))$ is B -stable. Thus there exists a T -invariant curve C in $X(w)$ such that $C^T = \{e, r_{\alpha+\beta}\}$ with weight $\alpha + \beta$ at e . Since $\alpha + \beta \in \Psi^-$, $C \subset G_\Psi/B_\Psi$ as well, so $C \subset Y(v)$. Consequently $\alpha + \beta$ is a weight of $TE(Y(v))$, hence $TE(Y(v))$ is B_Ψ -stable. Since $Y(v)$ is rationally smooth and Ψ is stellar, it suffices to verify that $Y(v)$ is smooth if Ψ is of type B_n for $n = 2, 3$ or C_3 . This can be checked directly, but it's more efficient to use the following lemma.

Lemma 2. *Theorem 2 holds when G is of type B or C .*

Proof. We have already verified this for type C . Thus let $X(w)$ be a rationally smooth Schubert variety in G/B such that $TE(X(w))$ is a B -submodule of $T_e(X(w))$, where G is type B . By the remarks in Section 3, it suffices to show $T_e(X(w)) = TE(X(w))$. since $\dim TE(X(w)) = \ell(w)$. If $X(w)$ is singular, then one can apply Billey's pattern avoidance criterion [1, Theorem 3]. Namely, w contains the pattern $\bar{2}\bar{1}$ in signed permutation notation. By the argument on p.113 of [3], it follows that there exists an $\alpha \in \Phi(w)$ and a $\beta > 0$ such that $\mathfrak{g}_{\alpha+\beta} \subset T_e(X(w))$ for which the inequality

$r_{\alpha+\beta} \leq w$ fails. This says that $\mathfrak{g}_\alpha \subset TE(X(w))$, while $\mathfrak{g}_{\alpha+\beta} \not\subset TE(X(w))$, contradicting the assumption that $TE(X(w))$ is B -stable. Thus $X(w)$ must be smooth at e , consequently smooth. \square

Theorem 2 now follows from Theorem 5 and the lemma. \square

Let us next prove Theorem 3. To establish (3), we note the following result [7, Theorem 2]: For any $x \leq w$, let $\Theta(w, x)$ denote the linear span of the reduced tangent cone to $X(w)$ at x . Let $\mathcal{H}(w, x)$ be the convex hull of $\Phi(w, x) = \{\alpha \in \Phi \mid x^{-1}(\alpha) < 0, r_\alpha x \leq w\}$. Then

$$\Theta(w, x) \subset \sum_{\alpha \in \mathcal{H}(w, x) \cap \Phi} \mathfrak{g}_\alpha,$$

and any $\gamma \in \mathcal{H}(w, x) \cap \Phi$ which isn't a T -weight of $\Theta(w, x)$ has the form $\gamma = \beta + \epsilon\mu$, where $\beta \in \Phi(w, x)$, $\mu > 0$, $\epsilon \in \{1, 2\}$ and $x^{-1}(\mu) < 0$. Thus, if $x = e$, γ cannot exist. \square

The fact that, in general, $\Theta(w)$ is the B -module span of $TE(X(w))$ is proved explicitly in Theorem 3 of [7].

5. TWO EXAMPLES

The first example shows that Theorem 2 fails without the G_2 hypothesis. That is, there exists a rationally smooth but singular Schubert variety $X(w)$ in G_2/B for which $TE(X(w))$ is a B -submodule of $\mathfrak{g}_2/\mathfrak{b}$. Recall that all Schubert varieties in G/B are rationally smooth when G has rank two.

Example 1. Let α and β denote respectively the negatives of long and short simple roots for G_2 corresponding to B , and let $r = r_\alpha$ and $s = r_\beta$ be the corresponding reflections. Let $w = srsrs$ and consider $X(w)$. Now $\ell(w) = 5$ and it is not hard to see that

$$\Phi(w) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta\}.$$

Thus $TE(X(w))$ is indeed a B -submodule of $T_e(X(w))$. However, it is well known that $X(w)$ is singular: for example, see [12]. \square

Example 2. In this example, we consider a singular Schubert variety in the flag variety $SO(5)/B$ of type B_2 . Let α and β denote respectively the negatives of the long and short simple roots as in the previous example, and let $w = srs$. We claim $\Phi(w) = \{\alpha, \beta, \alpha + 2\beta\}$, so $\Theta(w)$ is the B -module with weights $\{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$. Thus $TE(X(w)) \neq \Theta(w)$, hence, by Theorem 2, $X(w)$ is singular. Note that in the signed permutation notation for the elements of $W(B_2)$ (cf. [1]), $w = \bar{2}\bar{1}$. It is well known and easy to see that w is the unique element of $W(B_2)$ such that $X(w)$ is singular. \square

It would be interesting to determine which B -submodules of $\mathfrak{g}/\mathfrak{b}$ are tangent spaces at the identity to a smooth Schubert variety in G/B . This might lead to an efficient counting procedure for enumerating the smooth Schubert varieties.

References

- [1] S. Billey: *Pattern avoidance and rational smoothness of Schubert varieties*, Adv. Math. **139** (1998), no. 1, 141–156.
- [2] S. Billey and T. Braden: *Lower bounds for Kazhdan-Lusztig polynomials from patterns*, Transform. Groups **8** (2003), no. 4, 321–332.
- [3] S. Billey and V. Lakshmibai: *Singular loci of Schubert varieties*, Progress in Mathematics, **182** Birkhuser Boston, Inc., Boston, MA, 2000.
- [4] S. Billey and S. A. Mitchell: *Smooth and palindromic Schubert varieties in affine Grassmannians* J. Algebraic Combin. **31** (2010), no. 2, 169–216.
- [5] S. Billey and A. Postnikov: *Smoothness of Schubert varieties via patterns in root subsystems*. Adv. in Appl. Math. **34** (2005), no. 3, 447–466.
- [6] J. B. Carrell: *The Bruhat Graph of a Coxeter Group, a Conjecture of Deodhar, and Rational Smoothness of Schubert Varieties*, Proc. Symp. in Pure Math. **56**, No. 2, (1994), Part 1, 53–61.
- [7] J. B. Carrell: *The span of the tangent cone of a Schubert variety*, Algebraic groups and Lie groups, 51–59, Austral. Math. Soc. Lect. Ser., 9, Cambridge Univ. Press, Cambridge, 1997.
- [8] J. B. Carrell and J. Kuttler: *Singular points of T -varieties in G/P and the Peterson map*, Invent. Math. **151** (2003), 353–379.
- [9] C. Chevalley: *Sur les decompositions cellulaires des espaces G/B* , Proc. Symp. in Pure Math. **56** (1994), Part I, 1–25.
- [10] V. V. Deodhar: *Local Poincar duality and nonsingularity of Schubert varieties*, Comm. Algebra **13** (1985), no. 6, 1379–1388.
- [11] D. Kazhdan and G. Lusztig: *Schubert varieties and Poincar´ duality*. Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), 185–203, Proc. Sympos. Pure Math., **36**, Amer. Math. Soc., Providence, R.I., 1980.
- [12] S. Kumar: *Nil Hecke ring and singularity of Schubert varieties*, Inventiones Math., **123** (1996), 471–506.
- [13] V. Lakshmibai and B. Sandhya: *Criterion for smoothness of Schubert varieties in $SL(n)/B$* , Proc. Indian Acad. Sci. Math. Sci. **100** (1990) 45–52.
- [14] V. Lakshmibai and C. S. Seshadri: *Singular locus of a Schubert variety*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), no. 2, 363–366.
- [15] P. Polo: *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Indag. Math. (N.S.) **5** (1994), no. 4, 483–493.
- [16] K. M. Ryan: *On Schubert varieties in the flag manifold of $Sl(n, C)$* , Math. Ann. **276** (1987) 205–224.

James B. Carrell
 Department of Mathematics
 University of British Columbia
 Vancouver, Canada V6T 1Z2
 carrell@math.ubc.ca